Descartes

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We have consulted [Des54] and [Des] the first for the translation and facsimilie of the original and the second largely for the introduction which connects Descartes philosphy and geometry.

Between this document and the previous I had the pleasure of speaking with Christopher Hollings, I asked if there was anything between Diophantus and Descartes. Essentially the answer, which I already suspected, was that there was almost nothing. There is a couple of Italians *right* before Descartes, but similar to how Euclid captured and then surpassed the previous writters on geometry, Descartes can in a sense (less completely) be said to have summarised their advancements and then completed them. There were many arabic scholars between them however, whose works we will perhaps look at later.

Remark. Another interesting thing that came up in the discussion is the dualism, conflation and distinction between arithmetic and geometry. The point that is relevant here is that before the Greeks this was not a feature of mathematics. The Egyptians, Mesopotamians etc did mathematics (as far as we can tell) in a mostly unreflective way. They did not ponder the philosophical issues. Hence they conflated an area with the measurement of that area, the line with the length of the line, the geometric with the arithmetic. The Greeks perhaps did too much thinking and separated these two out, and it took many years before they could be put back together again by Descartes as we will see.

1 Arithmetic VS Geometry

Lets set the scene as I now see it. Sometime before Euclid (the precise time is to be decided by a deeper dive into the literature on Greek mathematics) geometry and arithmetic are seperated. Despite the similarities of the two sciences (as seen in Euclids axiomatisation) they are seperated *on principle*. Numbers can be mulitplied as many times as one likes, where as lines multiply to give planes, and planes and lines multiply to give volumes, this is the end of the line for geometric multiplication. Numbers come in discrete quantities or their inverses, lines are continuous and can be incommensurable with one another.

This is where Descartes enters the scene, with virtually no precursors. His geometry "begins where he (Vieta) left off" and contains "nothing in it that (he) believes to be known by anyone else". The first thing he does is clarify the situation left by the Greeks.

> the considerations that forced ancient writers to use arithmetical terms in geometry, thus making it impossible for them to proceed beyond a point where they could see clearly

the relation between the two subjects, caused much obscurity and embarrassment, in their attempts at explanation.

He then clarifies the relation between the two subjects which we quote at length.

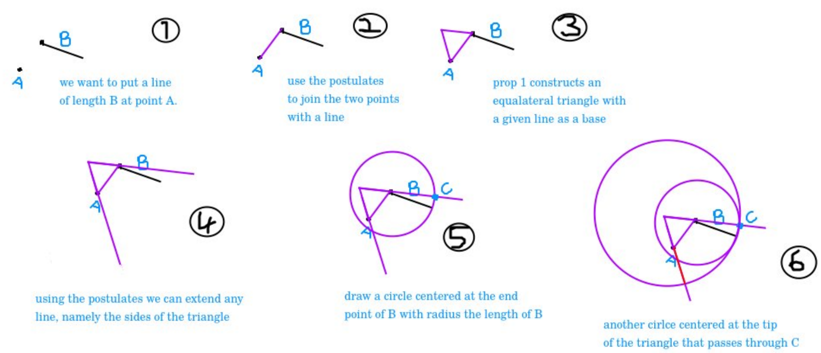
> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction. Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots ..., so in geometry, to find required lines it is merely necessary to add or subtract other lines ; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication) ; or, again, to find a fourth line which is to one of the given lines as unity is to the other (which is equivalent to division) ; or. finally, to find one, two, or several mean proportionals between unity and some other line (which is the same as extracting the square root, cube root, etc., of the given line. And I shall not hesitate to introduce these arithmetical terms into geometry, for the sake of greater clearness.

Here he is *defining* what he means by addition, multiplication etc of lines. He has thrown out the Greek method of multplication and introduced a notion which corresponds completely the the *length* of lines. Thus geometry and arithmetic are finally unified, arithmetic is about the *length* of the lines in geometry! We will now go through all of these constructions a bit more carefully (as Descartes does).

Addition Recall from [Euc08, Book I, Prop 2] that we have

> To place a straight-line equal to a given straight-line at a given point (as an extremity).

which is constructed as follows



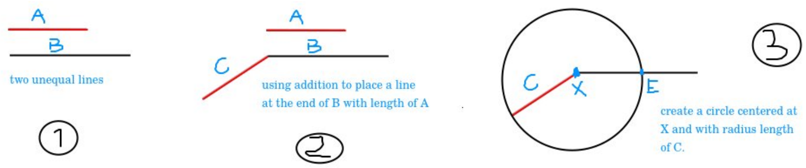
Then the argument is that the red line is the same length as B because: B is the same length as the line joining from the end of B to C which is the same length as the line joining the tip of the triangle to C minus the length of the side of the triangle, which is the same length as the line joining the tip of the triangle to the end of the red line (on the circle) minus the length of the triangle which is the red line (some of the axioms are used in this deduction, see Euclid for the details).

Now if we take A to be the end of *a different line* then we have shown that given two lines we may extend one by the other.

Subtraction Again from Euclid Book I, now proposition 3, we have

> For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

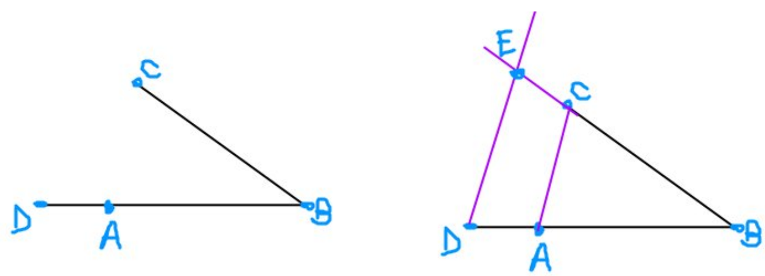
The construction is done as follows



and therefore the line connecting X and E is the segment of B that has the same length as A. This is subtracting A from B by considering the segment of the line that lies outside the circle.

Multiplication This is where Descarte comes into his own. His construction is as follows

> let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA ; then BE is the product of BD and BC.



Joining two points is a postulate, he then also uses that one can extend any given line. Note that he uses that one can draw parallel lines which is [Euc08, Book I, Prop 31]. Notice that if the two given lines are on the same line then this construction does not work. He does not explain *why* this construction works, as he is more or less making a definition. On the other hand "the sides of similar triangles are proportional" and one can see that if we consider AB to be length one then the length of DB must be $DB \times DA = DB \times 1 = DB$. This tells us that the proportion the sides of the constructed triangle are in to the original is DB and hence the line BE must be $DB \times BC$.

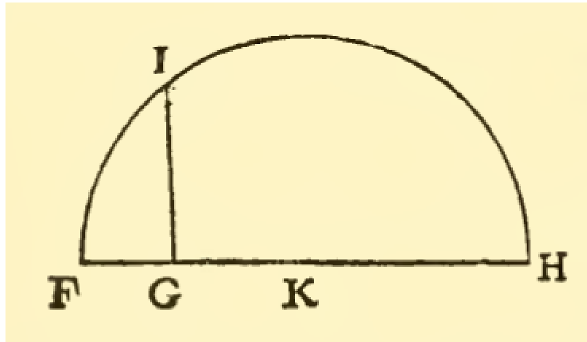
His division is exactly the same construction in reverse: Descarte writes

> If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE ; then BC is the result of the division.

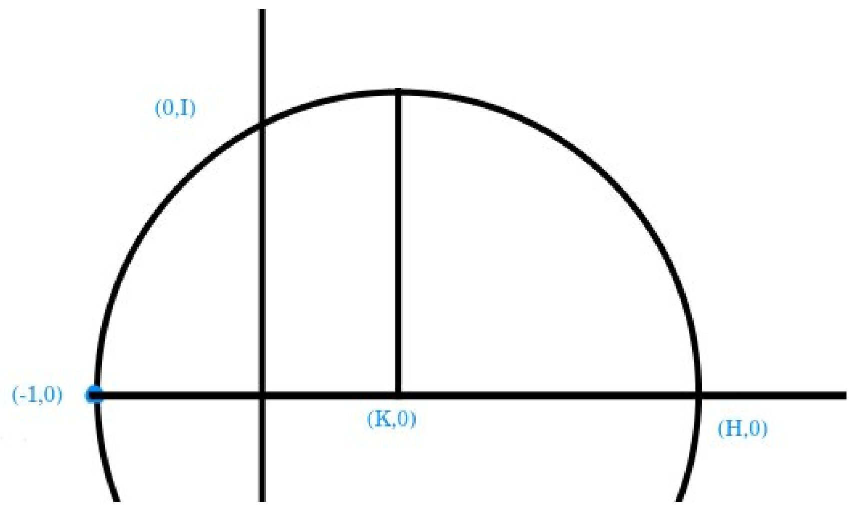
Root Extraction

> If the square root of GH is desired, I add, along the same straight line, EG equal to unity ; then, bisecting EH at K, I describe the circle EIH about K as a center, and draw from G a perpendicular and extend it to I, and GI is the required root. I do not speak here of cube root, or other roots, since I shall speak more conveniently of them later.

with the picture



Bisecting, extending lines and inscribing circles are all either postulates or demonstrated in Euclid, constructing a perpendicular is book I prop 12. Again this is a definition. Lets see in modern language a proof of this: First we redo the setup within the modern "Cartesian" plane



We are trying to show that $I = \sqrt{H}$. First we observe that we have the following three expressions for the radius of the circle

$$\text{radius} = H - K = K + 1 = \frac{H+1}{2}$$

using that it is centered at K and that it bisects the bottom line. Using our modern ideas we see that the equation of a circle with this radius centered at $(K, 0)$ is given by

$$(x - K)^2 + y^2 = (K + 1)^2$$

and because I is now just the intersection of this circle with the y axis we can solve for y when $x = 0$

$$K^2 + y^2 = (K + 1)^2 = K^2 + 2K + 1 \implies y^2 = 2K + 1 = H$$

and we are done.

Remark. Comparing [Smi] to Descartes is interesting. That article is much shorter and much less eloquent. It is moreover still in the old mode of geometry even though there is some sparks of inovation especially with regards to notation and algebraic manipulation of geometry, it fails to unify the subjects as Descartes does, keeping mulitplication for instance of two lines to be a plane.

Remark. Note that Descarte still rejects negative solutions to algebraic equations.

2 Notation and Axioms

Notation Descartes is the first apparently to use a,b,c for knowns and x,y,z for unknowns in equations. He uses $\propto$ for $=$.

Descartes in his mission to merge geometry and arithmetic also merges their notation. Before Descartes already the notation a^2 for the literal square given by multiplying a line by itsefl was already used, and in algebra there was the further notation of a^n which of course made no sense in geometry. Descartes definitions however allow this notation to be imported directly into geometry and this he does.

> I ordinarily mean only simple lines, which, however, I name squares, cubes, etc., so that I may make use of the terms employed in algebra.

Thus much of the notational advances that had recently taken hold in *algebra* were acquired by *geometry.* All the usual notation for addition $(+)$, multiplication (concatenation), subtraction $(-)$ and division $(numerator/denominator)$ as well as square roots $(\sqrt{})$ is already present in Descartes, the innovation is that previously they only made sense in algebra.

Axioms

> In the solution of a geometrical problem I take care, as far as possible, to use as lines of reference parallel lines or lines at right angles ; and I use no theorems except those which assert that the sides of similar triangles are proportional, and that m a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides.

In his dealing with curved lines Descartes notes that the Greeks failed to go beyond conics, circles and straight lines, in analysing the reason for this he writes

> I cannot believe, either, that it was because they did not wish to make more than two postulates, namely, (1) a straight line can be drawn between any two points, and (2) about a given center a circle can be described passing through a given point. In their treatment of the conic sections they did not hesitate to introduce the assumption that any given cone can be cut by a given plane. Now to treat all the curves which I mean to introduce here, only one additional assumption is necessary, namely, two or more lines can be moved, one upon the other, determining by their intersection other curves. This seems to me in no way more difficult.

Moreover he also gives a construction of a perpendicular line to a given curve (pg. 117). In general it seems that Descartes is working roughly within the axiomatic system of Euclid however he is much less careful in showing all the steps of his deductions, or where he deviates from it he is at least careful about what extra assumptions he is making.

Remark. We can see in this section the lie that the Cartesian plane was invented by Descartes, as he writes

> Probably the real explanation of the refusal of ancient geometers to accept curves more complex than the conic sections lies in the fact that the first curves to which their attention was attracted happened to be the spiral, the quadratrix, and similar curves, which really do belong only to mechanics, and are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination

where if he had the full power of the Cartesian plane he would have easily have been able to parametrise such a curve. Indeed it is clear long before this that he would have no conseption of a plane as pairs of coordinates or "real numbers". The real numbers as such did not exist, there were lengths of lines and even complex numbers from complex solutions to algebraic equations, but the (Cauchy) completion of these sets would not have made any sense. Really there was much more to go before arriving at the Cartesian plane as we have it now, which probably originates in Hilberts Geometry, but certainly after Cantor.

Remark. Descartes has some interesting remarks about the types of lines he sees as permissable to add to Geometry:

> Thus, no matter how we conceive a curve to be described, provided it be one of those which I have called geometric, it is always possible to find in this manner an equation determining all its points.

and that

> On the other hand, geometry should not include lines that are like strings, in that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact. Nevertheless, since strings can be used in these constructions only to determine lines whose lengths arc known, they need not be wholly excluded.

3 Analytic Geometry

This is the thing most often attributed to Descartes as inventing. From the the wikipedia on analytic geometry:

> In analytic geometry, any equation involving the coordinates specifies a subset of the plane, namely the solution set for the equation, or locus.

This is contrasted with *synthetic geometry*, wherein subsets of the plane were given as ruler and compass constructions or more generally from some set of axioms deduced to exist.

It is clear that Descartes does treat of shapes constructed from equations (or vice versa), this is his whole method. The modern treatment likes to mention things like "coordinates". In Descartes this is given by specifying a "unit", that is a sort of reference line. Descartes needs this to define his algebraic operations and therefore translate equations into a geometric operations. The $\mathbb{R}^2$ way of constructing a plane and labelling points however is very foriegn to Descartes, as we mentioned above, however the "coordinates" and "equations" methods have been translated there.

Remark. A more detailed comparison with the work of Fermat, for instance in French here or translated partially [Smi91, III. Fermat on Analytic Geometry], would be interesting. I tried to read this translation and could not really make out anything of interest, without more context. I admit a preference for Descartes style.

4 Example

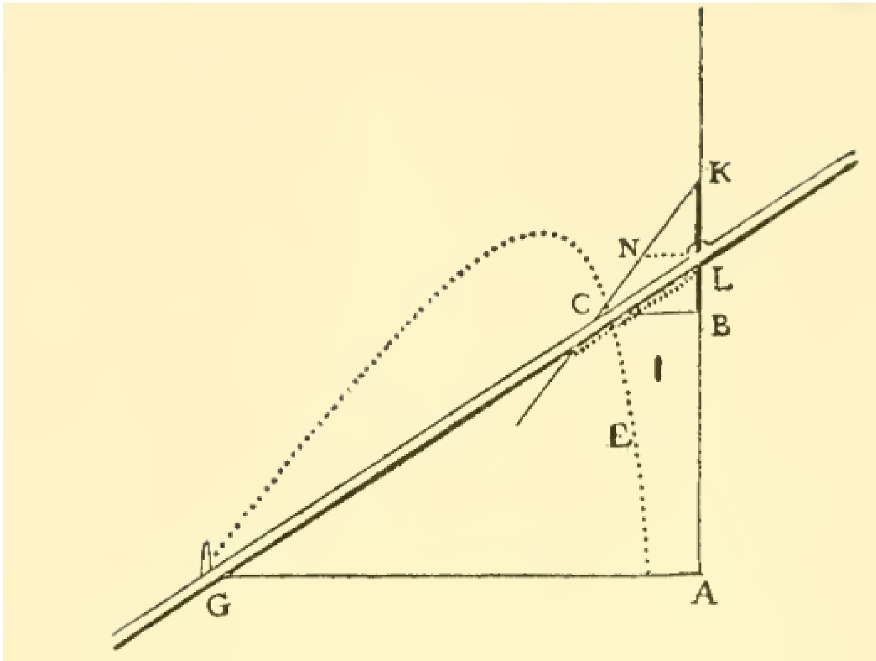
> For example, if I have $z^2 = az + b^2$,[21] I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to $\frac{1}{2}a$, that is, to half the other known quantity which was multiplied by z, which I supposed to be the unknown line. Then prolonging MN, the hypotenuse[22] of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way:[23]
>
> $$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}.$$

Curves. Descartes is somewhat terse, and complains of his tiring of writing, thus it is no surprise that many of the details are (explicitly) left to the reader. In the case of the construction of curves we have already seen that he beleives "it is always possible to find in this manner an equation determining all its points". Despite this, and maybe for a lack of my own careful reading, it is not as systematic as it sounds. These class of curves Descartes calls "geometric" and he describes it as follows

> all points of those curves which we may call "geometric." that is, those which admit of precise and exact measurement

He merely asserts that they "must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation." The converse is not clearly discussed in Descarte, that is going from an equation to a curve (although aparently it was in the article of Fermat referenced above). What we do get is an example of a translation of a problem in the plane to an equation. Let us first quote it at length and then attempt a sort of modern English translation:

> Suppose the curve EC to be described by the intersection of the ruler GL and the rectilinear plane figure CNKL, whose side KN is produced indefinitely in the direction of C, and which, being moved in the same plane in such a way that its side[52] KL always coincides with some part of the line BA (produced in both directions), imparts to the ruler GL a rotary motion about G (the ruler being hinged to the figure CNKL at L).[53] If I wish to find out to what class this curve belongs, I choose a straight line, as AB, to which to refer all its points, and in AB I choose a point A at which to begin the investigation.[54] I say "choose this and that," because we are free to choose what we will, for, while it is necessary to use care in the choice in order to make the equation as short and simple as possible, yet no matter what line I should take instead of AB the curve would always prove to be of the same class, a fact easily demonstrated.[55]



Then I take on the curve an arbitrary point, as C, at which we will suppose the instrument applied to describe the curve. Then I draw through C the line CB parallel to GA. Since CB and BA are unknown and indeterminate quantities, I shall call one of them y and the other x. To the relation between these quantities I must consider also the known quantities which determine the description of the curve, as GA, which I shall call a; KL, which I shall call b; and NL parallel to GA, which I shall call c. Then I say that as NL is to LK, or as c is to b, so CB, or y, is to BK, which is therefore equal to $\frac{b}{c}y$. Then BL is equal to $\frac{b}{c}y - b$, and AL is equal to $x + \frac{b}{c}y - b$. Moreover, as CB is to LB, that is, as y is to $\frac{b}{c}y - b$, so AG or a is to LA or $x + \frac{b}{c}y - b$. Multiplying the second by the third, we get $\frac{ab}{c}y - ab$ equal to

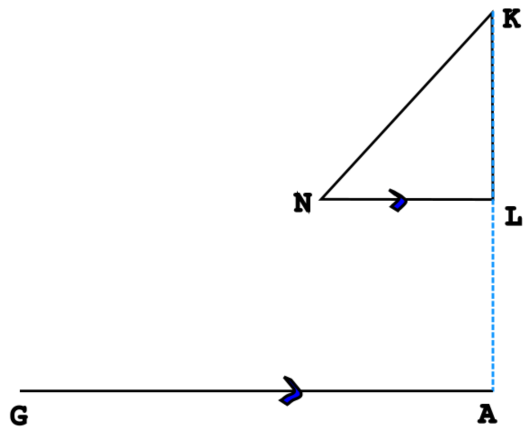
$$xy + \frac{b}{c}y^2 - by,$$

which is obtained by multiplying the first by the last. Therefore, the required equation is

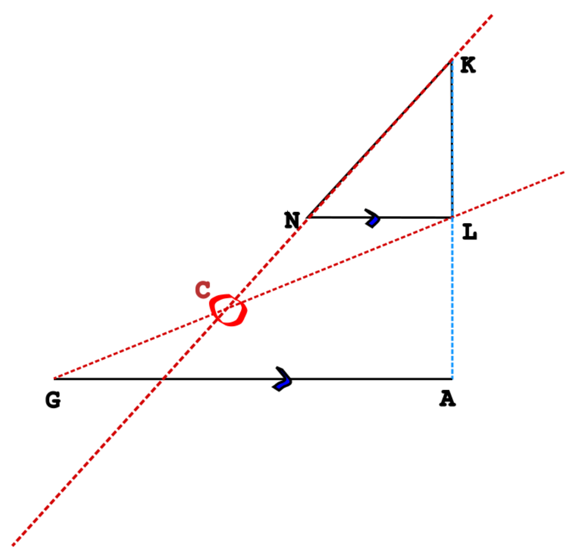
$$y^2 = cy - \frac{cx}{b}y + ay - ac.$$

One thing that we can see happening here is that Descarte begins with a figure and then applies a coordinate system over the top. For our translation we will do the same construction but just start with the "coordinates" first, to show the parity with modern methods.

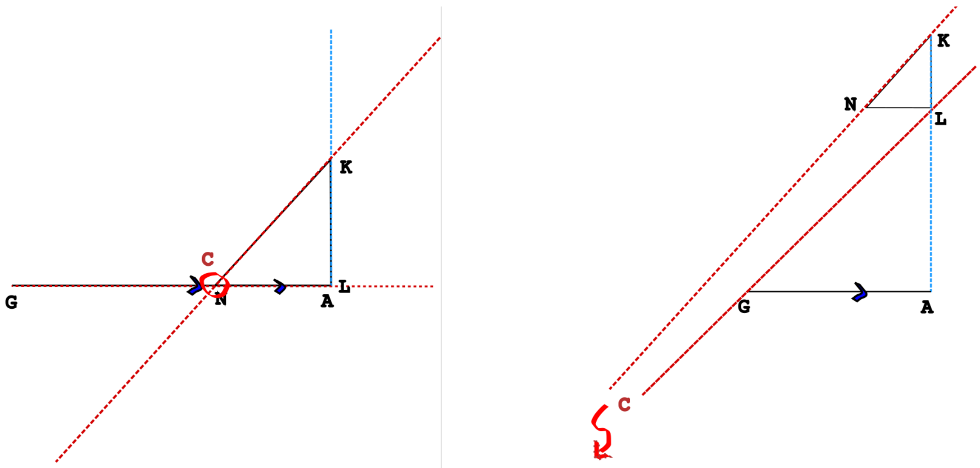
This document was most essential in trying to understand what Descarte means. First we fix a line (or two points) GA. Next we fix a triangle (or a tripple of points) KNL *such that* NL is parallel to GA and the line passing through KL also passes through A.



On this diagram we will draw two lines and the goal is to describe their intersection as we slide the triangle up and down the (infinite) dotted blue line (without changing its dimensions). The first line is that joining G and L, the second is the infinite line NK.

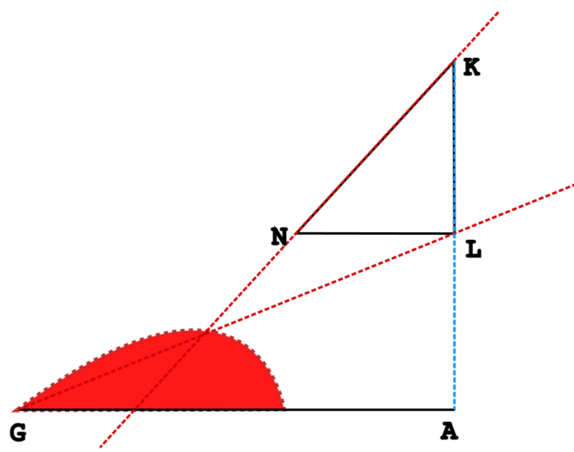


Looking at the extremes of the triangle slide can help image the curve that will result

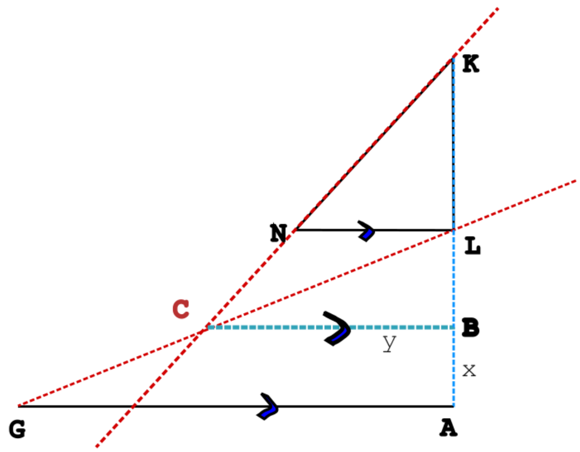


(a) At the bottom. (b) High up.

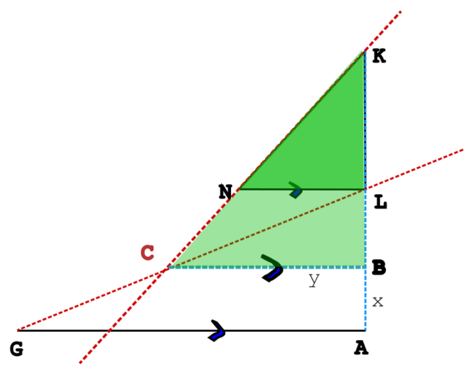
from which we can see that it traces out a curve that passes through a point distance N from A on GA, it passes through G and that it sort of goes off downward as the triangle tends upwards (the two lines approach being parallel).

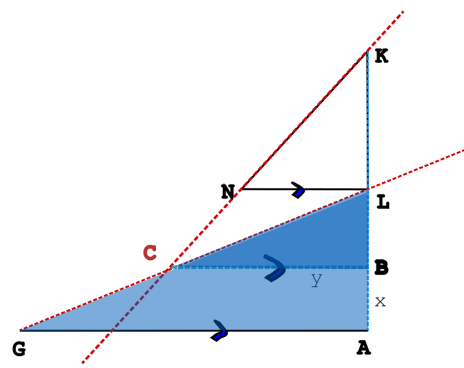


Finally the proof is effected by labeling certain edges and then comparing similar triangles, namely he labels the following:



that is he chooses a point B such that the line CB is parallel to GA and then labels its base y. This y is then the base of two different triangles, which have similar triangles, and from the fact that similar triangles have proportional sides is where he gets two equations for y and then he just substitues the first into the second.





Remark. This quote feels interesting but I dont know why:

> it requires two unknown quantities to express the relation between two points

Remark. From here

> An examination of his work shows that what he meant by this was any curve that could be drawn with a linkage (i.e. a device made of hinged rigid rods). Descartes' work indicates that he was well aware that this class of curves is exactly the class of all algebraic curves although he gave no formal proof of this. This theorem is scarcely known among modern mathematicians although it can be proved straight-forwardly by looking at linkages that add, subtract, multiply, divide and generate integer powers.

The reference given is I. I. Artobolevskii, Mechanisms for the Generation of Plane Curves, Macmillan, New York, 1964.

Remark. The above document summarises the relationship between Descarte and analytic geometry nicely as follows

> Descartes is touted to students today as the originator of analytic geometry but nowhere in the Geometry did he ever graph an equation. Curves were constructed from geometrical actions, many of which were pictured as mechanical apparatuses. After curves had been drawn Descartes introduced coordinates and then analyzed the curve-drawing actions in order to arrive at an equation that represented the curve. Equations did not create curves; curves gave rise to equations

The converse was discussed by Fermat and seems to have been understood more broadly at the time of Liebniz.

References

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